Math 222A Lecture 15 Notes

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October 19, 2021

1 Introduction to the Fourier Transform

1.1 Motivation: diagonalization for differential operators

We would like to have a better way to think about fundamental solutions to PDEs. Here is an analogy for the Fourier transform. Suppsoe we have a symmetric matrix in $\mathbb{R}^n$. Then A is diagonalizable, with orthonormal eigenvectors $u_1, \dots, u_n$. If you want to better represent your matrix, you can change coordinates to this basis, or you can express an arbitrary vector with $u = c_1 u_1 + \cdots c_n u_n$, where $c_j = u \cdot u_j$. If you have two (or a family of) commuting matrices, you can find an orthonormal basis of eigenvectors for both (or all) matrices simultaneously.

If we have PDEs with constant coefficients, then the operators $P(\partial), Q(\partial), \dots$ are all commuting operators. Can we find a common eigenbasis of functions? Here are some candidates for eigenfunctions $e^{ix\cdot\xi}$, where the i is there to make sure that these don't blow up at ∞. Then

$$P(\partial)e^{ix\cdot\xi} = P(i\xi)e^{ix\cdot\xi},$$

so these exponentials naively serve as eigenfunctions for these operators with eigenvalues $P(i\xi)$. Here, we don't always have real eigenvalues, but we have complex eigenvalues.

Here are some issues:

- Are these functions orthogonal? Consider the Hilbert space $L^2(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} |u|^2 \, dx < \infty$. If we consider the $L^2(\mathbb{R}^n)$ inner product, $u \cdot v = \int_{\mathbb{R}^n} u(x)v(x) \, dx$ (with v replaced by $\overline{v}$ for complex functions), are these orthonormal? In fact, $e^{ix\cdot\xi} \notin L^2$, so we cannot properly analyze

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi_1} e^{-ix\xi_2} \, dx.$$

- For our diagonaiization, we have uncountably many eigenvectors. $L^2(\mathbb{R}^n)$ is a separable Hilbert space with a countable orthonormal basis. So we have too many functions.

However, we can think of $e^{ix\cdot\xi}$ as *generalized eigenfunctions*. We can still ask the question: Given $f \in L^2(\mathbb{R}^n)$, can we write it as a superposition as $e^{ix\cdot\xi}$? That is, can we write

$$f(x) = \int e^{ix\cdot\xi} c(\xi)\, d\xi?$$

If we disregard the above issues, can we still recover an identity like $c_j = u \cdot u_j$ as before? We may want to try

$$c(\xi) = \int f(x) e^{-ix\xi}\, dx.$$

But since we have trouble normalizing the eigenfunctions, should there be a normalization constant in front?

If we can achieve such a representation, then we get a lot out of it:

$$P(\partial) f = \int e^{ix\cdot\xi} c(\xi) P(i\xi)\, d\xi.$$

So the map $f \mapsto P(\partial) f$ just acts diagonally on this basis: $c(\xi) \mapsto P(i\xi) \cdot c(\xi)$.

1.2 Properties of the Fourier transform

We will use the notation $D_j = \frac{1}{i}\partial_j$, so that $D_j e^{ix\cdot\xi} = \xi_j e^{ix\cdot\xi}$. So we will think of $P(D)$ instead of $P(\partial)$. In this notation, $P(D)e^{ix\cdot\xi} = P(\xi)e^{ix\cdot\xi}$, and we call $P(\xi)$ the **symbol** of P.

Example 1.1. If $P(x, D) = \sum_\alpha c_\alpha(x) D^\alpha$, then the symbol is $P(x, \xi) = \sum_\alpha c_\alpha(x)\xi^\alpha$.

Definition 1.1. The **Fourier transform** of a function f is

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

Our goal is to show that

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

For what f is $\widehat{f}$ well-defined? The integral is absolutely convergent if $f \in L^1$, i.e. $\int |f| < \infty$. We will not use L^1 functions much in our context. If we have $f \in L^1$, then

$$|\widehat{f}(\xi)| \leq \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1},$$

which we can write as

$$\|\widehat{f}\|_{L^\infty} \leq \frac{1}{(2pi)^{n/2}} \|f\|_{L^1}.$$

The problem is that we want to be able to undo the Fourier transform, and for L^∞ functions, the Fourier transform is not well-defined.

What about the Fourier transform on test functions? If $f \in \mathcal{D}$, then $\widehat{f} \in \mathcal{E}$, so there is no compact support. But if we have $f \in \mathcal{E}$, then $\widehat{f}$ does not exist, since the integral may not converge. It seems that $\mathcal{D}$ is too small, and $\mathcal{E}$ is too large. What should be our intermediate space where $\mathcal{F}$ acts? We will use the Schwartz space $\mathcal{S}$.[1] For $u \in \mathcal{S}$, we want the derivatives to not only be bounded but have decay at infinity.

Definition 1.2. The **Schwartz space** is the space of $C^\infty(\mathbb{R}^n)$ functions which are **rapidly decreasing**, in the sense that

$$|x^\alpha \partial^\beta u| \leq c_{\alpha,\beta}$$

for all $\alpha, \beta \in \mathbb{N}^n$.

The Schwartz space $\mathcal{S}$ is a locally convex space with seminorms

$$p_{\alpha,\beta}(u) = \|x^\alpha \partial^\beta u\|_{L^\infty}.$$

Theorem 1.1. *The Fourier transform is $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, and the inverse $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$.*

We have not proven that $(\mathcal{F}^{-1} f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}\, dx$ gives the inverse, but we will call it the inverse for now. How do we prove this theorem?

Observe that in the expression $x^\alpha \partial^\beta$, the order of x^α and ∂^β does not matter. How do ∂, x interact with the Fourier transform?

Proposition 1.1. *For $f \in \mathcal{S}$, $\partial_j \widehat{f} = -i\widehat{x_j f}$.*

Proof.

$$\begin{aligned}\partial_j \widehat{f}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} f(x)(-ix_j)\, dx \\ &= -i\widehat{x_j f}.\end{aligned}$$

□

Proposition 1.2. *For $f \in \mathcal{S}$, $\xi \widehat{f} = -i\widehat{\partial_x f}$.*

Proof.

$$\xi_j \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} f(x) \xi_j\, dx$$

Use integration by parts.

$$\begin{aligned}&= \frac{1}{(2\pi)^{n/2}} \int i\frac{\partial}{\partial x_j}(e^{-ix\cdot\xi}) f(x)\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int -i(e^{-ix\cdot\xi}) f(x)\, dx.\end{aligned}$$

□

[1] This is not the same as Schwarz from the Cauchy-Schwarz inequality. Professor Tataru got to meet Schwartz once.

So multiplication by x on the physical side is differentiation on the Fourier side, and multiplication by ξ on the Fourier side is differentiation on the physical side.

Proof. If $f \in \mathcal{S}$, then (using $\beta = 0$ and $|\alpha| \leq N$ for $N > n$)

$$|f(x)| \leq \frac{c_N}{(1+|x|)^N} \in L^1.$$

So $\|\widehat{f}\|_{L^\infty} \leq c\|f\|_{L^1}$.

Together, our propositions give us

$$\xi^\alpha \partial_\xi^\beta \widehat{f} = (-i)^{|\alpha|+|\beta|} \widehat{\partial_x^\alpha x^\beta f}.$$

Here, we have

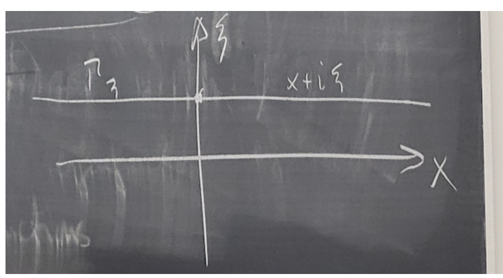
$$\|f^\alpha \partial_\xi^\beta \widehat{f}\|_{L^\infty} \leq \|\partial_x^\alpha x^\beta f\|_{L^1}.$$

If $f \in \mathcal{S}$, then $\partial_x^\alpha x^\beta f \in \mathcal{S} \subseteq L^1$. So the right hand side is finite, controlled by finitely many of our Schwartz seminorms. $\square$

Example 1.2 (Fourier transform of a Gaussian)**.** Suppose $f(x) = e^{-x^2/2}$. What is $\widehat{f}$?

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{(2\pi)^{n/2}} e^{-x^2/2} e^{-ix\xi}\, dx \\
&= \frac{1}{(2\pi)^{n/2}} e^{-\xi^2/2} \int e^{-(x+i\xi)^2/2}\, dx
\end{aligned}$$

How do we deal with this integral? If we write $z = x = i\xi$, we are doing a complex integral on the curve Γ_ξ:



So we get

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\Gamma_\xi} e^{-z^2/2}\, dz \\
&= e^{-\xi^2/2} \frac{1}{(2\pi)^{n/2}} \int_{\Gamma_0} e^{-z^2/2}\, dz \\
&= e^{-\xi^2/2} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-x^2/2}\, dx
\end{aligned}$$

We can recall $\int e^{-x^2}\,dx = \sqrt{\pi}$, so a change of variables gives

$$= e^{-\xi^2/2}.$$

So we have seen that

$$\mathcal{F}(e^{-x^2/2}) = e^{-\xi^2/2}.$$

In general, what is $\mathcal{F}(e^{-\lambda x^2/2})$? Here is how the Fourier transform behaves under scaling:

Proposition 1.3. *For $f \in \mathcal{S}$,*

$$\widehat{f(\mu\cdot)} = \frac{1}{\mu^n}\widehat{f}(\cdot/\mu).$$

Proof.

$$\mathcal{F}f(\mu x) = \int e^{-ix\cdot\xi} f(\mu x)\,dx$$

Make the change of variables $y = \mu x$.

$$= \frac{1}{\mu^n}\int e^{-iy\cdot\xi/\mu} f(y)\,dy$$

$$= \frac{1}{\mu^n}\widehat{f}(\xi/\mu).$$

□

Remark 1.1. You might call $f(\mu x)$ an L^∞ scaling, whereas $\frac{1}{\mu^n}\widehat{f}(\xi/\mu)$ is an L^1 scaling.

Example 1.3. Setting $\mu = \sqrt{\lambda}$,

$$\mathcal{F}(e^{-\lambda x^2/2}) = \frac{1}{\lambda^{n/2}} e^{-\xi^2/(2\lambda)}.$$

We will work towards the following Fourier inversion theorem:

Theorem 1.2. $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = I$ *in* $\mathcal{S}$.

Remark 1.2. You can think of $\mathcal{F}\mathcal{F}^{-1}$ as the complex conjugate of $\mathcal{F}^{-1}\mathcal{F}$.